University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2009
Mathematics 1101
Wednesday 14 January 2009 2.30 – 4.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1. (a) State what it means for a real sequence to converge.

(b) Prove that a convergent sequence is bounded.

(c) Show that if $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = m$ then $\lim_{n\to\infty}(x_n \cdot y_n) = l \cdot m$.

(d) Use the definition of convergence (not the combination theorem or other theorems) to show that

$$\lim_{n\to\infty} \frac{2n+1}{5n+3} = \frac{2}{5}.$$

2. (a) State the definition of $\lim_{x\to a^+} f(x) = l$.

(b) Let $f : (a,b) \to \mathbb{R}$ be a function and $\xi \in (a,b)$. Prove that

$$\lim_{x\to\xi} f(x) = l$$

if and only if, for each sequence $\langle x_n\rangle$ of points of (a,b) such that $x_n \neq \xi$, $n = 1, 2, \ldots$,

$$\lim_{n\to\infty} x_n = \xi \Longrightarrow \lim_{n\to\infty} f(x_n) = l.$$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 - x & (x \geq 1), \\ 2x & (x < 1). \end{cases}$$

Prove carefully (using ϵ and δ) that $f(x)$ is continuous at $x = 1$.

3. (a) State the Least Upper Bound Principle (continuum property).

(b) Find, if they exist, the sup, inf, max and min of the set

$$S = \{2^{-k} + 3^{-m} : k, m \in \mathrm{N}, k \geq 2\}.$$

(c) Define what it means for a sequence to be Cauchy.

(d) State the General Principle of Convergence.

(e) Show that if $\langle x_n\rangle$ is a convergent sequence, then it is a Cauchy sequence.

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4. (a) Prove that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

(b) State and prove the comparison test for two series

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} b_n$$

with non-negative terms.

(c) Determine with explanations whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}, \quad \sum_{n=1}^{\infty} \frac{n!(2n)!\sin(2n)}{(3n)!}, \quad \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}.$$

5. (a) State the Cauchy–Schwarz inequality.

(b) Show that, if $a_1, a_2, \ldots, a_n$ are all positive numbers, then

$$\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq n^2.$$

(c) State and prove the Intermediate Value Theorem.

(d) Assume $f : [0,1] \longrightarrow \mathbb{R}$ is continuous on the interval $[0,1]$ and $f(0) = f(1)$. Show that we can find a $c \in [0, 1/2]$ with

$$f(c) = f(c + 1/2).$$

6. (a) State the Arithmetic Mean – Geometric Mean Inequality for n non-negative numbers $a_1, a_2, \ldots, a_n$.

(b) Let $0 < a < b$. Define the sequence $\langle x_n \rangle$ by

$$x_1 = a, \quad x_2 = b, \quad x_{2n+1} = \sqrt{x_{2n}x_{2n-1}}, \quad x_{2n+2} = \frac{1}{2}(x_{2n} + x_{2n-1}).$$

Use induction to prove

$$x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n},$$

and deduce that the subsequences of even and odd subscripts converge to the same limit. What does this imply for the sequence $\langle x_n \rangle$?

END OF PAPER